Conditional Extremes

Find, if they exist, the conditional extremes of the following function: $f(x, y) = 6 - 4x - 3y$ Subject to: $x^2 + y^2 = 1$. Take into account the second order conditions.

Solution

We build the Lagrangian:

$$L = 6 - 4x - 3y + \lambda(1 - x^2 - y^2)$$

$$L'x = -4 - \lambda 2x = 0$$

$$L'y = -3 - \lambda 2y = 0$$

$$L'\lambda = 1 - x^2 - y^2 = 0$$

I solve for λ and equate:

$$\lambda = -4/2x$$

$$\lambda = -3/2y$$

$$-4/2x = -3/2y$$

$$4y/3 = x$$

I substitute into the third restriction:

$$1 - (4y/3)^2 - y^2 = 0$$

$$1 - 16y^2/9 - y^2 = 0$$

$$1 = 25y^2/9$$

$$9/25 = y^2$$

So that $y = 3/5$ or $y = -3/5$. If $y = 3/5$ then $x = (4/3)(3/5) = 4/5$. If $y = -3/5$, then $x = (4/3)(-3/5) = -4/5$. Therefore, we have two extremes: $(4/5, 3/5)$ and $(-4/5, -3/5)$.
Next, to check whether these points are maximum or minimum, we construct the bordered Hessian which has the following form: $\bar{H} = \begin{pmatrix} 0 & g'x & g'y \\ g'x & L''xx & L''xy \\ g'y & L''yx & L'yy \end{pmatrix}$ Where we have:

$$g'x = -2x$$

$$g'y = -2y$$

$$L''xx = -2\lambda$$

$$L''yy = -2\lambda$$

$$L''yx = L''xy = 0$$

The bordered Hessian then takes the following form:
$\bar{H} = \begin{pmatrix} 0 & -2x & -2y \\ -2x & -2\lambda & 0 \\ -2y & 0 & -2\lambda \end{pmatrix}$
We calculate the determinant of the bordered Hessian to check if it is positive or negative semi-definite, thus determining if we are dealing with a minimum, maximum, or saddle point:
The determinant of the bordered Hessian:

$$0[(-2\lambda)(-2\lambda)] - (-2x)[(-2x)(-2\lambda) - 0(-2y)] - 2y[(-2x)0 - (-2y)(-2\lambda)]$$

$$2x[4x\lambda] - 2y[-4y\lambda] = 8x^2\lambda + 8y^2\lambda$$

Evaluating at the first point $(4/5, 3/5)$ and considering that $\lambda = -4/2x = -5/2$.
The determinant of the Hessian is $8(4/5)^2(-5/2) + 8(3/5)^2(-5/2) < 0$. Therefore, we are dealing with a minimum.
Evaluating at the other point: (-4/5,-3/5) and considering that $\lambda = -4/2x = 5/2$. The determinant of the bordered Hessian is: $8(-4/5)^2(5/2) + 8(-3/5)^2(5/2) > 0$ Therefore, we are dealing with a maximum.